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## ARTICLES

# Critical finite-range scaling in scalar-field theories and Ising models

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We develop a critical finite-force-range scaling theory for $D$-dimensional scalar $\phi^n$ field theories that is based on a scaling ansatz equivalent to a Ginzburg criterion. To investigate its relationship to other scaling theories we derive equivalent results from renormalization groups and from finite-size crossover scaling for systems with weak long-range forces. By comparing our finite-range scaling relations with finite-*size* scaling relations for hypercylindrical systems above the upper critical dimension $D_c$, we arrive at a criterion of critical equivalence that provides an asymptotic mapping between the two kinds of systems. We apply our scaling relations to a $\phi^4$ Ginzburg-Landau Hamiltonian, to the one-dimensional Kac model with exponentially decaying interactions, and to the $N\times\infty$ quasi-one-dimensional Ising (Q1DI) model, in which each spin interacts with $O(N)$ others. Near the Gaussian mean-field critical point the Ginzburg-Landau Hamiltonians for all three models become identical, but for the Q1DI model this requires a length rescaling. For the Kac model the resulting scaling relations are those of a $D=1$ quartic field theory, and for the Q1DI model they are those of a cylindrical Ising system above $D_c$. Results of specialized numerical scaling techniques applied to transfer-matrix calculations for the Q1DI model with $N\leq 1024$ strongly support our theoretically obtained scaling relations.

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## I. INTRODUCTION

A number of physical systems display, for all practical purposes, mean-field critical behavior. Generally these are systems in which the interactions extend over distances much longer than the microscopic length scales. Examples include superconductors [1,2], magnets and binary mixtures with large but finite interaction ranges [3], and polymer mixtures with large but finite chain lengths [3–5]. It has long been known from theoretical arguments that models with weak, long-range interactions approach classical mean-field behavior as the interaction range approaches infinity while the interaction strength goes to zero [6–8]. In some such models renormalization-group calculations have indicated the presence of a "van der Waals" fixed point [9–12]. Several workers have proposed that the interaction range may be used as a finite parameter, in terms of which a phenomenological renormalization procedure valid near this mean-field critical point could be developed in analogy with the well-established finite-size scaling method [13–15]. Earlier studies along these lines concentrated on hypercubic systems in which all the particles interact equally strongly, and in which the finite scaling parameter is the total particle number [16,17]. Models that have more recently been investigated from this point of view include Ising chains with algebraically [18,19] or exponentially [20] decaying interactions.

The Kac model [7,21,22] is an Ising chain with exponentially decaying ferromagnetic interactions. Recently Privman compared its finite-range scaling behavior with the finite-size scaling of cylindrical Ising models above four dimensions [20]. He observed that whereas the critical exponents for both models take their classical values, and the scaling behaviors of the singular parts of their free energies are also identical, their correlation lengths scale differently with the interaction range. Part of the present paper is devoted to further elucidation of these results.

The outline of the remainder of this paper is as follows. In Sec. II we derive critical scaling relations for a general $D$-dimensional scalar $\phi^n$ field theory in terms of its interaction range, based on a scaling ansatz which we find is equivalent to a Ginzburg criterion [1–3,23]. By comparing these relations to finite-*size* scaling results for $L^{D-D'}\times\infty^{D'}$ "hypercylindrical" systems above their upper critical dimension we identify a condition of *critical equivalence,* under which the two kinds of systems can be asymptotically mapped onto one another at the mean-field critical point. In Sec. III these scaling relations are applied to three specific examples: a standard quartic Ginzburg-Landau (GL) Hamiltonian [2], the Kac model, and a particular long-range Ising chain system, the quasi-one-dimensional Ising (Q1DI) model [24–27]. We find the scaling relations for the Kac model to be identical to those for a one-dimensional Gaussian quartic field theory with weak long-range interactions, in agreement with Privman's conclusions [20], whereas the scaling relations for the Q1DI model agree with those expected for a cylindrical Ising system above its upper critical dimension [15]. These comparisons provide a specific example of the critical-equivalence mapping introduced in Sec. II. In Sec. IV we present numerical transfer-matrix results for the Q1DI model that strongly support the scaling relations derived in Sec. III. A method for filtering out corrections to scaling was developed, and a specialized convergence acceleration algorithm was used to obtain the highly precise numerical results.

The ambiguity of the notion of "length" in mean-field-like models is emphasized in some of the earlier treatments [16,17] of finite-range-scaling concepts. These treatments are discussed in Sec. V. The several ways in which finite-range scaling has been introduced indicate that it may be derived without relying on analogy with finite-*size* scaling. In this section we show that this is indeed the case by deriving scaling relations equivalent to those we obtained in Sec. II both from critical finite-size crossover scaling relations [5] and from renormalization-group calculations [9–11]. In Sec. VI we summarize our conclusions and discuss applications of the critical-equivalence mapping. We find a specific asymptotic mapping between a $D$-dimensional field theory and the Q1DI model, and we briefly discuss an application to the study of chain-length scaling in polymer blends [4,5].

## II. FINITE-RANGE SCALING OF A $\phi^n$ FIELD THEORY

We begin our study by developing a finite-range scaling (FRS) procedure for a general $\phi^n$ field theory with a quadratic gradient term and interactions of range $\mathcal{R}$, described by the GL Hamiltonian [2]

$$\mathcal{H}_n=\frac{1}{KT}\int_\Omega d^Dr[\tfrac{1}{2}\mathcal{R}^2(\nabla\phi)^2+V_n(\phi,t,h)] \ . \tag{1}$$

Here $D$ is the spatial dimensionality, $\Omega$ is the system volume, $\nabla$ is the $D$-dimensional gradient, $T$ is the absolute temperature, and $K$ is a constant. The local effective potential is

$$V_n(\phi,t,h)=-\tfrac{1}{2}t\phi^2+\frac{1}{n}\phi^n-\frac{n-2}{(n-1)^{(n-1)/(n-2)}}h\phi \ , \tag{2}$$

where the coefficients of the terms linear and quadratic in $\phi$ are the "field" and "temperature" variables $h$ and $t$, respectively. All quantities in Eqs. (1) and (2) are dimensionless. The partition function $\mathcal{Z}^\Omega$ is given as the functional integral

$$\mathcal{Z}^\Omega=C\int\mathcal{D}\phi e^{-\mathcal{H}_n(\phi)} \ , \tag{3}$$

where the constant $C$ contains the appropriate normalization factors.

The functional minimization required to obtain the equilibrium partition function leads by standard variational calculus to the Euler-Lagrange equation

$$-\mathcal{R}^2\nabla^2\phi-t\phi+\phi^{n-1}-\frac{n-2}{(n-1)^{(n-1)/(n-2)}}h=0 \ , \tag{4}$$

whose uniform equilibrium solution is the order parameter,

$$\bar\phi(t,h)=\begin{cases}\dfrac{n-2}{(n-1)^{(n-1)/(n-2)}}h|t|^{-\bar\gamma} & \text{for } t<0 \ , \ h\ll|t|^{(n-1)/(n-2)}\\[2mm] \dfrac{(n-2)^{1/(n-1)}}{(n-1)^{1/(n-2)}}h^{1/\bar\delta(n)} & \text{for } t=0\\[2mm] t^{\bar\beta(n)}+\dfrac{1}{(n-1)^{(n-1)/(n-2)}}ht^{-\bar\gamma'} & \text{for } t>0 \ , \ h\ll t^{(n-1)/(n-2)} \ ,\end{cases} \tag{5}$$

where the critical exponents all take their mean-field values,

$$\bar\gamma=\bar\gamma'=1 \ , \tag{6a}$$

$$\bar\delta(n)=n-1 \ , \tag{6b}$$

$$\bar\beta(n)=\frac{1}{n-2} \ . \tag{6c}$$

The coefficients in $V_n(\phi,t,h)$ of Eq. (2) are chosen so that $|\bar\phi(1,0)|=1$ for $n>2$, and also that for $t>0$ the stationarity condition $\partial V/\partial\phi=0$ has exactly one real solution for $|h|>h_{\rm sp}=t^{(n-1)/(n-2)}$, the mean-field "spinodal" field.

To obtain the correlation length $\xi$ in the Gaussian approximation, which by the Ginzburg criterion becomes exact in the limit $\mathcal{R}\to\infty$ [2], we consider $\phi(\mathbf{r})=\bar\phi[1+u(\mathbf{r})]$. Here $u(\mathbf{r})$ is a Gaussian random field with $\langle u(\mathbf{r})\rangle=0$ that satisfies the Euler-Lagrange equation,

$$[-\mathcal{R}^2\nabla^2-t+(n-1)\bar\phi^{n-2}]u(\mathbf{r})=0 \ , \tag{7}$$

whose Laplace transform yields

$$\xi^{-2}=\mathcal{R}^{-2}[(n-1)\bar\phi^{n-2}-t] \ . \tag{8}$$

Together with Eq. (5) for $\bar{\phi}$ this yields the scaling relation

$$\xi = \mathcal{R}|t|^{-\tilde{\nu}}\Lambda(h|t|^{-\tilde{\Delta}(n)}) \quad (9)$$

with the mean-field correlation-length and field-scaling exponents

$$\tilde{\nu} = \tfrac{1}{2} , \quad (10a)$$

$$\tilde{\Delta}(n) = \tilde{\beta}(n) + \tilde{\gamma} = \frac{n-1}{n-2} . \quad (10b)$$

[The scaling function $\Lambda(\zeta)$ is different for $t>0$ and $t<0$.]

The singular part of the free-energy density is $F = \Omega^{-1}[\mathcal{H}_n(\phi(\mathbf{r})) - \mathcal{H}_n(\bar{\phi})]$. Expanding $V_n(\phi(\mathbf{r}))$ to second order in $u$ and observing that integrals of odd powers of $u$ vanish, one obtains

$$F = \frac{\mathcal{R}^2\bar{\phi}^2\xi^{-2}}{\Omega KT}\int_\Omega d^D r[\tfrac{1}{2}\xi^2(\nabla u)^2 + \tfrac{1}{2}u^2] . \quad (11)$$

The gradient term $\xi^2(\nabla u)^2$ is of order unity on average, so the behavior of the integral is determined by the magnitude of the Gaussian fluctuations $u^2$. If $D$ is above the upper critical dimension $D_c$, these are always of order unity or less [2]. For $D<D_c$ the fluctuations are of order unity if the Ginzburg criterion [1–3,23] is satisfied, i.e., if $|t|/Z \geq O(1)$, where the Ginzburg parameter $Z \sim \mathcal{R}^{-D/\nu(D_c-D)}$ is the size of the nonclassical critical region [2]. In the limit $\mathcal{R}\to\infty$ the Ginzburg criterion is satisfied for all $t \neq 0$. Thus, both for $D<D_c$ and $D>D_c$ the integral in Eq. (11) is proportional to $\Omega$, so that

$$F \propto \frac{\mathcal{R}^2\bar{\phi}^2\xi^{D-2}}{\xi^D} . \quad (12)$$

We do not consider the special case $D=D_c$, for which logarithmic corrections may be present [15].

In analogy with the standard finite-size scaling (FSS) result for the relation between the free-energy density and the correlation length in $D$-dimensional cubic systems [15], we take as our FRS ansatz that

$$F \sim \xi^{-D} . \quad (13)$$

A discussion of the relations of this scaling ansatz to other finite-range and finite-size scaling hypotheses is given in Sec. V. In that section we also show how the scaling relations derived below alternatively can be obtained from renormalization-group results [9–11].

Combining Eqs. (5) and (9) for $\bar{\phi}$ and $\xi$ with Eqs. (12) and (13), we obtain, both for $t>0$ and $t<0$, the field rescalings

$$|t| \sim \mathcal{R}^{-D/\{\tilde{\nu}[D_c(n)-D]\}} , \quad (14a)$$

$$|h| \sim \mathcal{R}^{-D\tilde{\Delta}(n)/\{\tilde{\nu}(D_c(n)-D]\}} , \quad (14b)$$

with the $n$-dependent upper critical dimension

$$D_c(n) = \frac{2n}{n-2} . \quad (14c)$$

For $1<D<D_c(n)$ these relations are equivalent to the Ginzburg criterion with $\nu=\tilde{\nu}=\frac{1}{2}$. The resulting FRS relations for $F$ and $\xi$ are

$$F = \mathcal{R}^{-DD_c(n)/[D_c(n)-D]}\Phi(\tau,\zeta) , \quad (15a)$$

$$\xi = \mathcal{R}^{D_c(n)/[D_c(n)-D]}\Xi(\tau,\zeta) , \quad (15b)$$

with the scaling variables

$$\tau = |t|\mathcal{R}^{D/\{\tilde{\nu}[D_c(n)-D]\}} , \quad (15c)$$

$$\zeta = |h|\mathcal{R}^{D\tilde{\Delta}(n)/\{\tilde{\nu}[D_c(n)-D]\}} . \quad (15d)$$

In their studies of finite-*size* scaling for general $m$-vector models in a "hypercylindrical" $L^{D-D'}\times\infty^{D'}$ geometry with $D>D_c$ Singh and Pathria [28,29] obtain a scaling relation for the correlation length $\hat{\xi}$ near criticality and in zero external field. (The caret is introduced in our notation for the correlation length in the hypercylindrical geometry for reasons that will be made clear below.) This relation can be written as

$$\hat{\xi} = L^{(D-D')/[D_c(n)-D']}\hat{\Xi}(|t|L^{(D-D')/\{\tilde{\nu}[D_c(n)-D']\}}) , \quad (16)$$

and has been shown to agree with explicit results for a number of different models with O($m$) symmetry, including the Ising ($m=1$) model [30], general O($m$) models with $m\geq 2$ [31,32], and the spherical ($m\to\infty$) model [29,33,34].

Since Eqs. (15a)–(15d) and Eq. (16) represent the scaling behavior of two different systems that both asymptotically approach mean-field critical behavior, it seems natural to seek a mapping such that critical scaling relations derived for one system can be applied to the other. We therefore propose as a criterion of *critical equivalence* that the singular parts of the correlation lengths and of the free-energy densities of the two systems must be asymptotically proportional. In order to implement this mapping we define the range-normalized correlation length for the field theory, $\hat{\xi}\sim\xi/\mathcal{R}$, and we require the critical scaling relations for the two systems to agree with one another. This yields a relation between the interaction range $\mathcal{R}$ in the $D$-dimensional field theory and the length $L$ in the hypercylindrical system with infinite system size in $D'$ dimensions:

$$\mathcal{R}^{D/[D_c(n)-D]} \sim L^{(D-D')/[D_c(n)-D']} \sim N^{1/[D_c(n)-D']} , \quad (17)$$

where $N\sim L^{D-D'}$ is the hypercylinder cross section. Thus the scaling relations of Eqs. (15a)–(15d) give

$$F = L^{-(D-D')D_c(n)/[D_c(n)-D']}\Phi(\tau,\zeta)$$
$$= N^{-D_c(n)/[D_c(n)-D']}\Phi(\tau,\zeta) , \quad (18a)$$

$$\hat{\xi} = L^{(D-D')/[D_c(n)-D']}\hat{\Xi}(\tau,\zeta)$$
$$= N^{1/[D_c(n)-D']}\hat{\Xi}(\tau,\zeta) , \quad (18b)$$

$$\tau = |t|L^{(D-D')/\{\tilde{\nu}[D_c(n)-D']\}}$$
$$= |t|N^{1/\{\tilde{\nu}[D_c(n)-D']\}} , \quad (18c)$$

$$\zeta = |h|L^{\tilde{\Delta}(n)(D-D')/\{\tilde{\nu}[D_c(n)-D']\}}$$
$$= |h|N^{\tilde{\Delta}(n)/\{\tilde{\nu}[D_c(n)-D']\}} . \quad (18d)$$

## III. THREE SPECIFIC EXAMPLES

In this section we apply the FRS relations obtained in Sec. II to three different models that can be mapped onto a $\phi^4$ field theory: the standard GL Hamiltonian for a second-order phase transition, the Kac model, and the Q1DI model.

### A. Standard $\phi^4$ Ginzburg-Landau Hamiltonian

The conventional $\phi^4$ GL Hamiltonian [2,10,11,35] for the critical properties of a system with interaction range $R$ in a $D$-dimensional volume $\Omega$ is

$$\mathcal{H}_{\rm GL}=\frac{T_c}{T}\int_\Omega d^D r\left[\tfrac{1}{2}R^2(\nabla\psi)^2-\tfrac{1}{2}\left[1-\frac{T}{T_c}\right]\psi^2+\frac{1}{4}\frac{K}{T_c}\psi^4-\frac{H}{T_c}\psi\right] . \quad (19)$$

In Eq. (19) $T$ is the temperature in energy units and $T_c$ is the mean-field critical temperature. The external field $H$ and the positive nonlinearity parameter $K$ also have dimensions of energy, whereas both $R$ and $r$ are dimensionless, expressed in units of a fixed lattice constant.

In order to facilitate the application of the FRS formalism developed in Sec. II, we express all energies in units of $T_c$ by redefining $T/T_c \to T$, $K/T_c\to K$, and $H/T_c\to H$. The zero-temperature order parameter is $|\psi_0|=K^{-1/2}$, and the zero-temperature spinodal field, for which both the first and second derivatives of the integrand in $\mathcal{H}_{\rm GL}$ vanish at the same value of $\psi$, is $H_{\rm sp0}=(2/3\sqrt{3})K^{-1/2}$. In terms of the normalized variables $\phi=\psi/|\psi_0|$ and $h=H/H_{\rm sp0}$ the GL Hamiltonian of Eq. (19) becomes

$$\mathcal{H}_{\rm GL}=\frac{1}{KT}\int_\Omega d^D r[\tfrac{1}{2}R^2(\nabla\phi)^2+V_4(\phi,t,h)] , \quad (20)$$

where

$$V_4(\phi,t,h)=-\tfrac{1}{2}t\phi^2+\tfrac{1}{4}\phi^4-(2/3\sqrt{3})h\phi . \quad (21)$$

Here $t=(1-T)$, and $V_4(\phi,t,h)$ is the local temperature-dependent effective potential of Eq. (2) with $n=4$.

By comparing Eqs. (20) and (21) with Eqs. (1) and (2), one sees that this case corresponds to the $D$-dimensional $\phi^n$ field theory of Sec. II with $n=4$ and $\mathcal{R}=R$. Thus Eqs. (15a)–(15d) become

$$F=R^{-4D/(4-D)}\Phi(\tau,\zeta) , \quad (22a)$$

$$\xi=R^{4/(4-D)}\Xi(\tau,\zeta) , \quad (22b)$$

$$\tau=|t|R^{2D/(4-D)} , \quad (22c)$$

$$\zeta=|h|R^{3D/(4-D)} . \quad (22d)$$

### B. Kac model

The Kac model [7] consists of a chain of Ising spins $s_i=\pm 1$ with exponentially decaying ferromagnetic pair interactions. It is described by the reduced Hamiltonian [7,21]

$$\mathfrak{H}_{\rm K}=-\frac{J}{2RT}\sum_{i\neq j}s_i s_j e^{-|i-j|/R}-\frac{H}{T}\sum_j s_j . \quad (23)$$

Closely following Kac's original treatment [7], we now derive the GL Hamiltonian for this model near its critical point.

The partition function for a chain of $I$ spins with periodic boundary conditions $(I+1\equiv 1)$ can be written, in the limit of large $I$, as

$$Z^I=(2e^{-J/(2RT)})^I\int_{-\infty}^{+\infty}d^I x\, h(x_1)\times\prod_{i=1}^{I}\mathcal{K}_R(x_i,x_{i+1})h(x_{I+1}) , \quad (24a)$$

where

$$h(x)=(2\pi e^{x^2})^{-1/4}\cosh^{1/2}\left[\left[\frac{J}{RT}\right]^{1/2}+\frac{H}{T}\right] , \quad (24b)$$

and the transfer kernel is

$$\mathcal{K}_R(x,y)=e^{1/2R}e^{-(1/2)U_R(x)}\times\frac{\exp\left[-\frac{(x-y)^2}{4\sinh(R^{-1})}\right]}{[4\pi\sinh(R^{-1})]^{1/2}}e^{-(1/2)U_R(y)} , \quad (24c)$$

in which

$$U_R(x)=\tfrac{1}{2}\tanh\left[\frac{x^2}{2R}\right]-\ln\cosh\left[\left[\frac{J}{RT}\right]^{1/2}+\frac{H}{T}\right] . \quad (24d)$$

(Note that the integration variables $x_i$, which play the role of a "local order parameter" are *continuous* effective molecular fields, regardless of the value of $R$ [7,35].) The partition function can be rewritten as

$$Z^I=\left[\frac{2e^{-(J-T)/(2RT)}}{[4\pi\sinh(R^{-1})]^{1/2}}\right]^I\times\int_{-\infty}^{+\infty}d^I x\, h(x_1)e^{-\mathcal{H}_{\rm K}(\{x_i\})}h(x_{I+1}) , \quad (25a)$$

where the GL Hamiltonian is

$$\mathcal{H}_{\rm K}(\{x_i\})=\sum_{i=1}^{I}\left[\frac{(x_{i+1}-x_i)^2}{4\sinh(R^{-1})}+U_R(x_i)\right] . \quad (25b)$$

The critical temperature for this model is $T_c=2J$ [7]. Again, we express $T$ and $H$ in units of $T_c$ and expand $U_R(x)$ for small $x$, small $H/T$, small $t$, and large $R$. The resulting quartic potential yields the zero-temperature order parameter $|x_0|=\sqrt{6R}$ and the spinodal field $H_{\rm sp0}=\frac{2}{3}$. Changing variables to $\phi=x/|x_0|$ and $h=H/H_{\rm sp0}$, and taking the spatial continuum limit $\sum_{i=1}^{I}\to\int_0^I dr$, we obtain the continuous GL Hamiltonian for the Kac model,

$$\mathcal{H}_{\rm K}=\frac{3}{T}\int_0^I dr[\tfrac{1}{2}R^2(\nabla\phi)^2+V_4(\phi,t,h)] . \quad (26)$$

The effective Hamiltonian $\mathcal{H}_K$ of Eq. (26) is identical to the quartic $\mathcal{H}_{GL}$ of Eq. (20) with $K=\frac{1}{3}$ and $D=1$. Therefore the Kac model has the same scaling behavior near criticality as a one-dimensional quartic field theory with interaction range $R$. (A similar derivation of a GL Hamiltonian for a $D$-dimensional Ising spin system is contained in Appendix B of Ref. [35].) Thus Eqs. (15a)–(15d) become

$$F=R^{-4/3}\Phi(\tau,\zeta) , \tag{27a}$$

$$\xi=R^{4/3}\Xi(\tau,\zeta) , \tag{27b}$$

$$\tau=|t|R^{2/3} , \tag{27c}$$

$$\zeta=|h|R , \tag{27d}$$

in agreement with Privman's results [20].

### C. Q1DI model

The Q1DI model, which has been used in transfer-matrix studies of metastability [24–27], consists of a one-dimensional chain of $I$ subsystems, each of which contains $N$ Ising spins $s_{n,i}=\pm 1$. Each spin interacts with each of the $2N$ spins in the adjoining subsystems with interaction constant $J_1/N \neq 0$. Each spin can also interact with each of the $N-1$ other spins in its own subsystem with interaction constant $J_2/N$. The reduced Hamiltonian is

$$\mathfrak{H}_Q=-\frac{NJ_1}{T}\sum_{i=1}^{I} m_i m_{i+1}-\frac{NJ_2}{2T}\sum_{i=1}^{I} m_i^2 -\frac{NH}{T}\sum_{i=1}^{I} m_i+\frac{J_2}{2T} , \tag{28}$$

which contains only the subsystem magnetizations $m_i=N^{-1}\sum_{n=1}^{N} s_{n,i}$. The partition function can be written [24,25] as

$$Z^I=\sum_{\{m_i\}} e^{-\mathcal{H}_Q(\{m_i\})} , \tag{29a}$$

where

$$\mathcal{H}_Q=\sum_{i=1}^{I}\mathcal{H}_Q(m_i,m_{i+1}) =\sum_{i=1}^{I}\left[\frac{NJ_1}{2T}(m_{i+1}-m_i)^2+U_N(m_i)\right] , \tag{29b}$$

and the local potential is

$$U_N(m_i)=-\frac{N(2J_1+J_2)}{2T}m_i^2-\ln\begin{bmatrix} N \\ \frac{N}{2}(m_i+1)\end{bmatrix} -\frac{NH}{T}m_i+\frac{J_2}{2T} , \tag{29c}$$

with the binomial coefficient giving the multiplicity of a state of magnetization $m_i$. Expanding $U_N$ for small $m_i$, small $H/T$, and large $N$, we find $T_c=2J_1+J_2$. Expressing $T$ and $H$ in units of $T_c$ as before, and expanding to lowest order in $t$, we obtain a potential quartic in $m$, which yields the zero-temperature order parameter $|m_0|=\sqrt{3}$ and spinodal field $H_{sp0}=\frac{2}{3}$. Changing variables to $\phi=m/|m_0|$ and $h=H/H_{sp0}$ and removing constant terms, we find that $U_N(m)\approx(3N/T)V_4(\phi,t,h)$, where $V_4(\phi,t,h)$ is the effective quartic potential of Eq. (21). In taking the continuum limit in the transfer direction we allow an $N$-dependent scaling factor $a(N)$ for lengths in this direction so that $(\phi_{i+1}-\phi_i)^2\to a^2(N)(\nabla\phi)^2$ and $\sum_{i=1}^{I}\to[1/a(N)]\int_0^{Ia(N)}dr$. The resulting GL Hamiltonian is

$$\mathcal{H}_Q=\frac{3N}{Ta(N)}\int_0^{Ia(N)}dr[\tfrac{1}{4}(1-\tilde{J}_2)a^2(N)(\nabla\phi)^2 +V_4(\phi,t,h)] , \tag{30}$$

where $\tilde{J}_2=J_2/T_c$.

The effective Hamiltonian $\mathcal{H}_Q$ of Eq. (30) can be made identical to $\mathcal{H}_n$ of Eq. (1) with $n=4$, $K=\frac{1}{3}$, $D=1$, and the interaction range $\mathcal{R}=(1/\sqrt{2})(1-\tilde{J}_2)^{1/2}N$, by choosing $a(N)=N$. The correlation length $\hat{\xi}$, in units of the original lattice constant, as it is usually calculated from the transfer-matrix spectrum [36], is therefore related to the correlation length $\xi$ of the corresponding field theory as $\hat{\xi}=\xi/N$. This is in agreement with the definition $\hat{\xi}\sim\xi/\mathcal{R}$ given in our discussion of the critical-equivalence mapping in Sec. II. Thus from Eqs. (15a)–(15d) we have

$$F=(1-\tilde{J}_2)^{-2/3}N^{-4/3}\Phi(\tau,\zeta) , \tag{31a}$$

$$\hat{\xi}=(1-\tilde{J}_2)^{2/3}N^{1/3}\hat{\Xi}(\tau,\zeta) , \tag{31b}$$

$$\tau=|t|(1-\tilde{J}_2)^{1/3}N^{2/3} , \tag{31c}$$

$$\zeta=|h|(1-\tilde{J}_2)^{1/2}N . \tag{31d}$$

In addition to the finite-range scaling, these relations also predict the dependence of the dominant eigenvalues of the transfer matrix on the intrasubsystem interaction $\tilde{J}_2$. As pointed out by Privman [20], this $N$-scaling behavior is in agreement with predictions for an $L^{D-1}\times\infty$ cylindrical Ising system of dimension $D>D_c$ and cross section $N\sim L^{D-1}$. Equations (31a)–(31d) are analogous to Eqs. (18a)–(18d) and represent a special case of the critical-equivalence mapping connecting the bulk FRS of a $D$-dimensional field theory and the FSS of a hypercylinder above $D_c$, which we discussed at the end of Sec. II.

## IV. NUMERICAL TRANSFER-MATRIX RESULTS FOR THE Q1DI MODEL

In this section we test by numerical transfer-matrix calculations our scaling predictions, Eqs. (31a)–(31d), for the $N\times\infty$ Q1DI model. This test is performed by inserting in Eqs. (15a)–(15d) the appropriate values of $\mathcal{R}$, $D$, and the length rescaling factor $a$ given at the end of Sec. III C, while leaving $D_c$, $\nu$, and $\Delta$ as "unknown" parameters to be numerically determined and compared with the predicted mean-field values, $D_c(4)=4$, $\tilde{\nu}=\frac{1}{2}$, and $\tilde{\Delta}(4)=\frac{3}{2}$. The predicted scaling relations in explicit form are

$$F_N=(1-\tilde{J}_2)^{-D_c/[2(D_c-1)]}N^{-D_c/[D_c-1]}\Phi(\tau_N,\zeta_N)\ , \quad (32a)$$

$$\hat{\xi}_N=(1-\tilde{J}_2)^{D_c/[2(D_c-1)]}N^{1/[D_c-1]}\hat{\Xi}(\tau_N,\zeta_N)\ , \quad (32b)$$

with the scaling variables

$$\tau_N=|t|(1-\tilde{J}_2)^{1/[2\nu(D_c-1)]}N^{1/[\nu(D_c-1)]}\ , \quad (32c)$$

$$\zeta_N=|h|(1-\tilde{J}_2)^{\Delta/[2\nu(D_c-1)]}N^{\Delta/[\nu(D_c-1)]}\ , \quad (32d)$$

where the $N$-dependence has been explicitly emphasized through subscripts. (The $\sqrt{2}$ factors from $\mathcal{R}$ have been absorbed in the scaling functions.) As discussed in detail below, we find that the resulting estimates are consistent to high precision with the predicted mean-field values, as well as with each other.

Numerical values for $F_N$ and $\hat{\xi}_N$ were obtained from a standard transfer-matrix calculation [36] analogous to those performed in Refs. [24–27]. The partition function of Eq. (29a) can be written as $\mathcal{Z}_N^I=\mathrm{Tr}\{\mathbf{T}_N^I\}$, where the $(N+1)\times(N+1)$ transfer matrix $\mathbf{T}_N$ is defined by its matrix elements, $\langle m_i|\mathbf{T}_N|m_{i+1}\rangle=\exp[-\overline{\mathcal{H}}_Q(m_i,m_{i+1})]$, and $\overline{\mathcal{H}}_Q(m_i,m_{i+1})$ is identical to $\mathcal{H}_Q(m_i,m_{i+1})$ of Eq. (29b), except that it is symmetrized with respect to $m_i$ and $m_{i+1}$ for computational convenience. The rank of $\mathbf{T}_N$ is $N+1$ for $\tilde{J}_2<1$, and it is 1 for $\tilde{J}_2=1$. In the usual fashion $F_N$ and $\hat{\xi}_N$ are given in the limit $I\to\infty$ by the two largest eigenvalues of $\mathbf{T}_N$, $\lambda_0$ and $\lambda_1$, as $F_N^{\mathrm{TM}}=-(T/N)\ln\lambda_0$ and $\hat{\xi}_N=[\ln(\lambda_0/|\lambda_1|)]^{-1}$. The transfer-matrix free energy $F_N^{\mathrm{TM}}$ contains a regular part that must be removed in order to isolate the singular part $F_N$. The transfer matrix was computed for finite subsystems up to $N=1024$ at the critical point, $T_c=2J_1+J_2$ and $h=0$, and then was tridiagonalized by the Householder method. The dominant eigenvalues were iteratively extracted from the tridiagonal matrix. All computations were performed in 64-bit precision on a Cray Y-MP/432 supercomputer in approximately five hours of CPU time. The high precision in the results was deemed sufficient for the present study, although greater precision may be achieved by adding results from system sizes as large as 2048 at a cost of about 20 CPU hours or by diagonalizing the transfer matrix in 128-bit precision at a cost of about 40 CPU hours.

For each quantity $P_N$ for which the bulk ($N$-scaling) exponent was sought, a sequence of values $\{P_N\}_{N=1}^{1024}$ was generated from transfer-matrix calculations for $N\times\infty$ systems with $\tilde{J}_2=0$ fixed. Given such a sequence, a function

$$W_N(\{P_N\})=(\ln 2)^{-1}\ln(P_{2N}/P_N)\ , \quad (33)$$

may be constructed in order to isolate the bulk exponent. For a sequence $\{P_N\}$ that is expected to represent a series of the form

$$P_N=A_0N^{\omega_0}+A_1N^{\omega_1}+A_2N^{\omega_2}+\cdots\ , \quad (34)$$

where $\omega_0>\omega_1>\cdots$, the function $W_N(\{P_N\})$ provides

$$W_N(\{P_N\})=\omega_0+O(N^{\omega_1-\omega_0})\ . \quad (35)$$

Two methods are used in this work to improve the convergence with $N$ of the sequence $\{W_N\}$. Both methods are discussed briefly below, and further detail is provided elsewhere [37].

The first method is a transformation on $\{P_N\}$ that filters out corrections to scaling. If $\{P_N\}$ is a data sequence whose behavior is described by Eq. (34), a sequence $\{P_N'\}$ with half as many elements as $\{P_N\}$ may be constructed by assigning

$$P_N'=2^{-\omega'}P_{2N}-P_N\ , \quad (36)$$

so that if $\omega'=\omega_1+\delta_1$ for $|\delta_1|\ll 1$, then

$$P_N'=A_0'N^{\omega_0}+A_1'N^{\omega_1}+A_2'N^{\omega_2}+\cdots\ , \quad (37)$$

where $A_1'=O(\delta_1)$. The function $W_N$ is then calculated as in Eq. (33) with $P_N'$ instead of $P_N$. Successive applications of the filter to lower-order terms generates sequences $\{P_N^{(j)}\}$ of the form

$$P_N^{(j)}=A_0^{(j)}N^{\omega_0}+\sum_{i=1}^{j}A_i^{(j)}N^{\omega_i}+O(N^{\omega_j+1})\ , \quad (38)$$

in which $A_i^{(j)}=O(\delta_i)$ where $1\le i\le j$.

If we suppose that coefficient $A_1$ in Eq. (34) is not constant, but is instead $O((\ln N)^p)$, then $A_1'=O((\ln N)^{p-1})$, so that the convergence of $\{W_N\}$ would not improve as a result of subsequently filtering lower-order terms. On the other hand, if each $A_i$ is constant, then the convergence of $W_N$ can improve exponentially as the filter is applied to each successive correction term. The successive filtering process continues either until the remaining corrections to scaling are comparable either with one or more of the terms $A_i^{(j)}N^{\omega_i}$, or with the numerical noise in $P_N$, or until the successive sequences reduce to the two points required to compute $W_1$. The effects of this filtering are illustrated with our data in Figs. 1 and 2.

The second method used to accelerate the sequence $\{W_N\}$ is a transformation that has been successful in accelerating various types of logarithmically convergent sequences with very few available elements [38]. This is the Levin $u$ transform, which generates a family of sequences $\{W_{k,n}\}_{n=1}^{N-k}$, $k=0,1,2,\ldots,N-1$, whose last elements are estimates for $\lim_{N\to\infty}W_N$:

$$W_{k,N-k}=\frac{\Delta^k[(N-k+1)^{k-2}W_{N-k}/\Delta(W_{N-k-1})]}{\Delta^k[(N-k+1)^{k-2}/\Delta(W_{N-k-1})]}\ , \quad (39a)$$

where

$$\Delta^k(S_N)=\begin{cases}S_N & \text{for } k=0\\ \Delta^{k-1}(S_{N+1})-\Delta^{k-1}(S_N) & \text{for } k\ge 1\ ,\end{cases} \quad (39b)$$

and we take $W_0=0$. If the numerical noise in $\{W_N\}$ is small, the estimates $W_{k,N-k}$ reach a plateau where they vary little with $k$, and the values on this plateau are averaged to give a final estimate for the bulk exponent.

Filters such as that in Eq. (36) were applied to each physical variable obtained from the transfer-matrix calculation in order to remove terms of order $N^{\omega_0-m/3}$, $m=1,2,3,\ldots$, where $\omega_0$ is the bulk exponent sought.

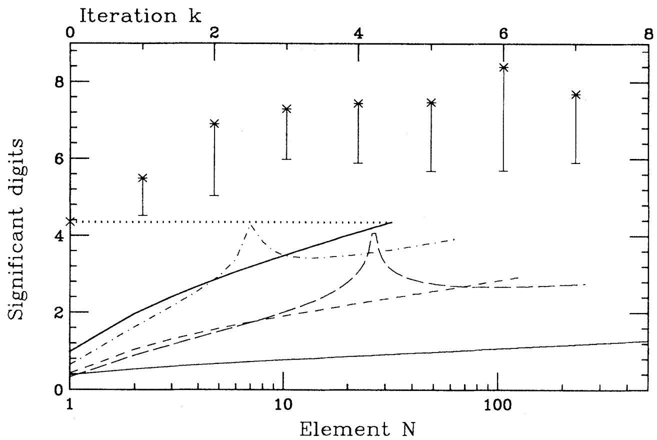


FIG. 1. The agreement between the finite-$N$ estimates $W_N(\{\hat{\xi}_N\})$ to the bulk exponent of $\hat{\xi}_N$ and the value $(\frac{1}{3})$ predicted by mean-field theory, given in significant digits. The thin solid line represents estimates from the raw transfer-matrix data. The long-dashed, short-dashed, dot-dashed, and thick solid lines represent the same data after filters have been applied successively to $\{\hat{\xi}_N\}$ to remove terms of order $N^0$, $N^{-1/3}$, $N^{-2/3}$, and $N^{-1}$, respectively. (Note: the spikes that appear on these curves are a result of the estimates crossing the mean-field value, which is due to the presence of alternating terms.) The points "×" indicate the agreement for the $u$-transform estimates $W_{k,N-k}$ for $\lim_{N\to\infty} W_N$, averaged over four subsets of the final sequence. The iteration index $k$ is shown on the top axis, and the ends of the error bars opposite the points indicate the significant digits of estimated error. The horizontal dotted line serves as a guide to the eye, linking the $k=0$ estimate to the last element in the final sequence.

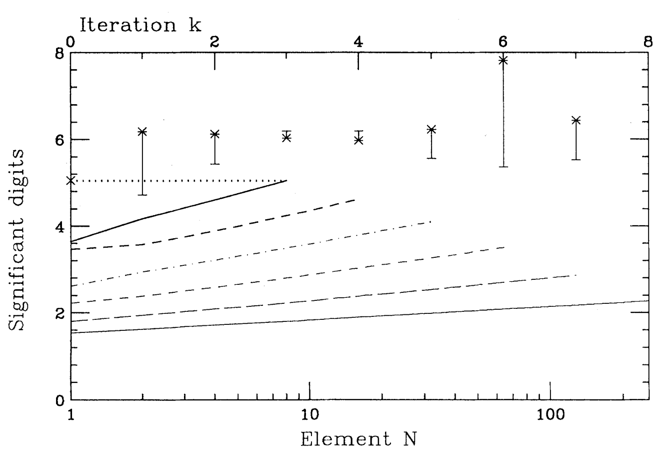


FIG. 2. The agreement between the finite-$N$ estimates $W_N(\{F_N\})$ to the bulk exponent of $F_N$ and the value $(-\frac{4}{3})$ predicted by mean-field theory, given in significant digits. This figure is analogous to Fig. 1. The long-dashed, short-dashed, dot-dashed, thick dashed, and thick solid lines represent the transfer-matrix data after filters have been applied successively to remove terms of order $N^{-5/3}$, $N^{-2}$, $N^{-7/3}$, $N^{-8/3}$, and $N^{-3}$, respectively. The points "×" indicate the agreement for the $u$-transform estimates $W_{k,N-k}$ for $\lim_{N\to\infty} W_N$, calculated from the final sequence.

The process continued until the error estimates for the bulk exponents stopped decreasing. Up to that point, the convergence of $\{W_N\}$ improved exponentially with each stage of filtering, indicating that the dominant corrections to scaling are extremely close to the orders given above, and that there are no logarithmic corrections apart from the bulk term. When the bulk term was filtered, $\{W_N\}$ converged to the first correction term, indicating that even the bulk term does not contain logarithmic corrections. This agrees with observations in earlier FRS studies [16–19], and represents a marked contrast to the situation in finite-*size* scaling of short-range force systems at $D_c$ [15], as discussed in Sec. V. Where sufficiently precise numerical data were available, the sequences $\{W_N\}$ were extrapolated by the $u$ transform. In cases where more than eight elements were available after filtering, the sequences were divided into "independent" eight-element subsequences, and the results of the $u$ transform were averaged over these subsequences.

The estimates $W_N(\{\hat{\xi}_N\})$ for the bulk exponent of the correlation length of Eq. (32b) with $\tilde{J}_2=0$ were calculated using the filter described above to remove terms to $O(N^{-1})$. These estimates were extrapolated by the $u$ transform to $0.333\,333\,3(15)$, giving $D_c=4.000\,000(14)$, in excellent agreement with the predicted mean-field value $D_c(4)=4$. The effects on $W_N$ of successive filtering of $\{\hat{\xi}_N\}$ and the further improvement resulting from the extrapolation are illustrated in Fig. 1. The numerical values are shown, together with our complete $N$-scaling results, in Tables I and II. To test the consistency of the bulk exponent estimate for $\hat{\xi}_N$, as well as the predicted relation between the $N$-dependence of $F_N$ and $\hat{\xi}_N$, the bulk exponent of the singular free-energy density $F_N$ of Eq. (32b) was estimated. The first two terms of the nonsingular part, a constant $-\ln 2$ and a term of $O(N^{-1})$, were removed from the transfer-matrix free energy $F_N^{\mathrm{TM}}$ in order to isolate the bulk term of the singular part $F_N$. The estimates $W_N(\{F_N\})$ for the bulk exponent were calculated with terms to $O(N^{-3})$ filtered. The resulting sequence was extrapolated to $-1.333\,34(3)$, giving $D_c=3.999\,9(2)$, in very good agreement with the result obtained from $\hat{\xi}_N$. The results of successive filtering and extrapolation on $W_N(\{F_N\})$ are shown in Fig. 2. Both Figs. 1 and 2 clearly show the exponential improvement of the estimates brought about by filtering.

Next, estimates were obtained for the scaling exponents $\nu^{-1}$ and $\Delta/\nu$ in the usual way [39] from derivatives of $\hat{\xi}_N$ and $F_N$. Differentiating both sides of Eq. (32b) with respect to $t$ once or with respect to $h$ twice, one finds expressions that can be tested numerically:

$$\frac{\partial \hat{\xi}_N(t,h)}{\partial t} = (1-\tilde{J}_2)^{(D_c+\nu^{-1})/[2(D_c-1)]} \times N^{(1+\nu^{-1})/(D_c-1)} \hat{\Xi}_t(\tau_N,\zeta_N)\,, \tag{40a}$$

$$\frac{\partial^2 \hat{\xi}_N(t,h)}{\partial h^2} = (1-\tilde{J}_2)^{(D_c+2\Delta/\nu)/[2(D_c-1)]} \times N^{(1+2\Delta/\nu)/(D_c-1)} \hat{\Xi}_h(\tau_N,\zeta_N)\,. \tag{40b}$$

The scaling exponents $\nu^{-1}$ and $\Delta/\nu$ were obtained from

TABLE I. Estimates of the bulk ($N$-scaling) exponents for the physical variables of Eqs. (32a), (32b), and (40a)–(41b). Each power of $N$ filtered is indicated, as is the use of the $u$ transform. Each bulk exponent is shown in terms of the mean-field scaling exponents with its predicted value.

| Physical variable | Mean-field bulk exponent | Powers of $N$ filtered | $u$ transform | Calculated bulk exponent |
|---|---|---|---|---|
| $\hat{\xi}_N$ | $\frac{1}{D_c(4)-1}=\frac{1}{3}$ | $0,-\frac{1}{3},-\frac{2}{3},-1$ | yes | 0.333 333 3(15) |
| $F_N$ | $-\frac{D_c(4)}{D_c(4)-1}=-\frac{4}{3}$ | $0,-1,-\frac{5}{3},-2,-\frac{7}{3},-\frac{8}{3},-3$ | yes | −1.333 34(3) |
| $\frac{\partial\hat{\xi}_N}{\partial t}$ | $\frac{1+\tilde{\nu}^{-1}}{D_c(4)-1}=1$ | $\frac{2}{3},\frac{1}{3},0,-\frac{1}{3},-\frac{2}{3},-1,-\frac{4}{3}$ | yes | 1.000 005(12) |
| $\frac{\partial^2\hat{\xi}_N}{\partial h^2}$ | $\frac{1+2\tilde{\Delta}(4)/\tilde{\nu}}{D_c(4)-1}=\frac{7}{3}$ | $2,\frac{5}{3},\frac{4}{3}$ | no | 2.334(10) |
| $(c_H)_N$ | $\frac{2\tilde{\nu}^{-1}-D_c(4)}{D_c(4)-1}=0$ | $-\frac{1}{3},-\frac{2}{3}$ | no | 0.002(3) |
| $(\chi_T)_N$ | $\frac{2\tilde{\Delta}(4)/\tilde{\nu}-D_c(4)}{D_c(4)-1}=\frac{2}{3}$ | $\frac{1}{3},0,-\frac{1}{3}$ | no | 0.666 6(2) |

the sequences $\{\partial\hat{\xi}_N/\partial t\}$ and $\{\partial^2\hat{\xi}_N/\partial h^2\}$ using these relations with $D_c=4$ as verified above. The exponents found were also in good agreement with the expected mean-field values and are shown in Tables I and II. In the case of the $h$ derivative numerical noise, attributable to roundoff error in the transfer-matrix calculation, precluded extrapolation. The estimate for the bulk exponent in this case was computed by averaging over the latter half of the finite-$N$ estimates $W_N$.

The scaling relation (32a) for $F_N$ can be differentiated with respect to $t$ twice or with respect to $h$ twice to obtain a relation for the specific heat or for the susceptibility, respectively:

$$(c_H)_N=(1-\tilde{J}_2)^{(2\nu^{-1}-D_c)/[2(D_c-1)]}\times N^{(2\nu^{-1}-D_c)/(D_c-1)}\mathcal{C}(\tau_N,\zeta_N)\ , \tag{41a}$$

$$(\chi_T)_N=(1-\tilde{J}_2)^{(2\Delta/\nu-D_c)/[2(D_c-1)]}\times N^{(2\Delta/\nu-D_c)/(D_c-1)}\mathcal{X}(\tau_N,\zeta_N)\ . \tag{41b}$$

The exponents $\nu^{-1}$ and $\Delta/\nu$ were calculated from these relations as well, again using $D_c=4$. As can be seen in Tables I and II, the exponents found were in good agreement with the results above, as well as with the expected mean-field values. In both cases numerical noise was substantial, so the bulk exponents were calculated by averaging over the latter half of the finite-$N$ estimates $W_N$. The estimates $W_N(\{(\chi_T)_N\})$ with $\{(\chi_T)_N\}$ filtered to $O(N^{-1/3})$ are shown in Fig. 3, giving an example of estimates that could not be extrapolated.

TABLE II. Estimates of the scaling exponents as determined from the calculated bulk exponents shown in Table I. The exact mean-field values for the scaling exponents are also shown.

| Scaling exponent | Mean-field value | Physical variable | Calculated value |
|---|---|---|---|
| $D_c$ | $D_c(4)=4$ | $\hat{\xi}_N$ | 4.000 000(14) |
| | | $F_N$ | 3.999 9(2) |
| $\nu^{-1}$ | $\tilde{\nu}^{-1}=2$ | $\frac{\partial\hat{\xi}_N}{\partial t}$ | 2.000 01(4) |
| | | $(c_H)_N$ | 2.000 3(5) |
| $\Delta/\nu$ | $\tilde{\Delta}(4)/\tilde{\nu}=3$ | $\frac{\partial^2\hat{\xi}_N}{\partial h^2}$ | 3.001(15) |
| | | $(\chi_T)_N$ | 2.999 9(3) |

To examine the $\tilde{J}_2$ dependence of the quantities $\hat{\xi}_N$, $F_N$, $\tau_N$, and $\zeta_N$, two sets of finite-$N$ data were calculated. For each quantity $P_N$, a set $\{P_N\}$ was calculated with $\tilde{J}_2=0$, and a set $\{P_N^*\}$ was calculated with $\tilde{J}_2=\frac{1}{2}$. If the bulk term in $P_N$ is

$$P_N\sim(1-\tilde{J}_2)^{v}N^{\omega_0}\ , \tag{42}$$

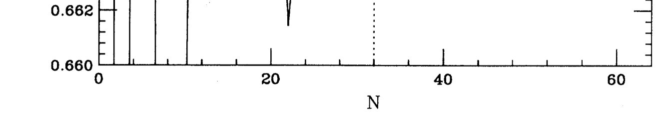


FIG. 3. The finite-$N$ estimates $W_N(\{(\chi_T)_N\})$ to the bulk exponent of $(\chi_T)_N$, after filtering of terms of order $N^{1/3}$, $N^0$, and $N^{-1/3}$ giving an example of the substantial numerical noise present in some of the data obtained. These estimates could not be extrapolated, so an average was taken over the latter half of the data. The cutoff is indicated by the vertical dotted line. The horizontal dashed line and error bar indicate the average estimate of 0.666 6(2) with its error. This estimate is very close to the mean-field theoretical prediction of $\frac{2}{3}$.

TABLE III. Estimates for the $(1-\tilde{J}_2)$ exponents for the physical variables of Eqs. (32a), (32b), (40a), and (41b). Each power of $N$ filtered is indicated as is the use of the $u$ transform. Each exponent is shown in terms of the mean-field scaling exponents with its predicted value.

| Physical variable | Mean-field $(1-\tilde{J}_2)$ exponent | Powers of $N$ filtered | $u$ transform | Calculated $(1-\tilde{J}_2)$ exponent |
|---|---|---|---|---|
| $\hat{\xi}_N$ | $\frac{D_c(4)}{2[D_c(4)-1]}=\frac{2}{3}$ | $0,-\frac{1}{3},-\frac{2}{3},-1,-\frac{4}{3},-\frac{5}{3}$ | yes | 0.666 67(3) |
| $F_N$ | $-\frac{D_c(4)}{2[D_c(4)-1]}=-\frac{2}{3}$ | $0,-1,-\frac{5}{3},-2,-\frac{7}{3}$ | yes | −0.666 69(7) |
| $\frac{\partial\hat{\xi}_N}{\partial t}$ | $\frac{D_c(4)+\tilde{v}^{-1}}{2[D_c(4)-1]}=1$ | $\frac{2}{3},\frac{1}{3},0,-\frac{1}{3},-\frac{2}{3},-1,-\frac{4}{3}$ | yes | 1.000 002(13) |
| $(\chi_T)_N$ | $\frac{2\tilde{\Delta}(4)/\tilde{v}-D_c(4)}{2[D_c(4)-1]}=\frac{1}{3}$ | $\frac{1}{3},0,-\frac{1}{3}$ | no | 0.333(6) |

then we expect $P_N=2^v P_N^*$ asymptotically. A function $W_N'$ can be constructed in analogy with Eq. (33):

$$W_N'(\{P_N\},\{P_N^*\})=(\ln 2)^{-1}\ln(P_N/P_N^*)\ , \qquad (43)$$

which asymptotically approaches the exponent $v$ for $(1-\tilde{J}_2)$ in the scaling relation.

The estimates $W_N'(\{\hat{\xi}_N\},\{\hat{\xi}_N^*\})$ for the correlation length were calculated with terms to $O(N^{-5/3})$ filtered. These estimates extrapolate by the $u$ transform to 0.666 67(3), in very good agreement with the predicted value of $\frac{2}{3}$. These values are presented along with our complete $\tilde{J}_2$-scaling results in Tables III and IV. The estimates $W_N'(\{F_N\},\{F_N^*\})$ for the singular free energy were calculated with terms to $O(N^{-7/3})$ filtered. The resulting sequence was extrapolated by the $u$ transform to −0.666 69(7), again in good agreement with the predicted value of $-\frac{2}{3}$. To test the $\tilde{J}_2$ dependence of the scaling variables $\tau_N$ and $\zeta_N$, the $(1-\tilde{J}_2)$ exponents for $\partial\hat{\xi}_N/\partial t$ and $(\chi_T)_N$ were estimated and compared with the result for $\hat{\xi}_N$ and $F_N$, respectively. As is shown in Tables III and IV, these exponents agree very well with the mean-field predictions. The fact that the $\tilde{J}_2$-scaling exponents of $F_N$ and $\hat{\xi}_N$ are found to be simply related by a change of sign, whereas the $N$-scaling exponents are not, confirms the correctness of the length rescaling used in the derivation of the scaling relations for $F$ and $\hat{\xi}$ in Sec. III C. A more detailed study of the critical finite-range scaling for the Q1DI model, including corrections to scaling, is reported elsewhere [37].

## V. ALTERNATIVE APPROACHES TO FINITE-RANGE SCALING

In this section we discuss several ways in which the FRS ansatz of Eq. (13), or, equivalently, the scaling relations of Eqs. (14a)–(15d), can be obtained. We also consider the relationship of finite-range scaling to the traditional finite-*size* scaling method and to crossover scaling [5] and renormalization-group results [9–11]. In addition, we discuss in more detail the possibility of logarithmic corrections to the FRS relations.

First we mention some different ways in which FRS ideas have previously been discussed in the literature.

(1) In studies of infinitely correlated systems in which neither length nor spatial dimensionality are well-defined concepts, Botet *et al.* [16,17] introduced the concept of a "coherence number" $\mathcal{N}\sim|t|^{-vD_c}$. By combining Eqs. (14a) and (15a) one finds that $F^{-1}$ has the same $t$-scaling behavior, and may thus be interpreted as a coherence number, independently of $D$.

(2) For $D=1$, results analogous to those obtained in Sec. II were found by Uzelac and Glumac [18,19] in the context of an Ising chain with algebraically decaying ferromagnetic interactions. Their approach was based on an analogy with FSS assumptions.

(3) Privman's results for the one-dimensional Kac model [20] were obtained by a rescaling of the Schrödinger equation resulting from a Legendre transform of the integrand in Eq. (25a).

(4) If the numerator in $F$, as given in Eq. (12), is viewed as the total free-energy cost of a critical fluctuation, as suggested by Unger and Klein [40], then our scaling ansatz becomes equivalent to a requirement that this cost should be independent of the interaction range at the mean-field critical point.

Among these four approaches to FRS, only the second one makes an explicit appeal to the analogy with the usual FSS formalism. The several ways in which the scaling relations can be obtained indicate that mean-field FRS is in fact not just a direct consequence of FSS. We provide support for this conclusion by discussing next two different approaches that lead to the same scaling relations as our present one.

For $D>1$ the infinite system has a finite-temperature critical point even for finite-range interactions. In this case the field rescalings, Eqs. (14a) and (14b), that result

TABLE IV. Estimates for the $(1-\tilde{J}_2)$ exponents for the scaling variables $\tau_N$ and $\zeta_N$. Exact mean-field predictions are also shown.

| Scaling variable | Mean-field $(1-\tilde{J}_2)$ exponent | Calculated $(1-\tilde{J}_2)$ exponent |
|---|---|---|
| $\tau_N$ | $\frac{1}{2\tilde{v}[D_c(4)-1]}=\frac{1}{3}$ | 0.333 33(3) |
| $\zeta_N$ | $\frac{\tilde{\Delta}(4)}{2\tilde{v}[D_c(4)-1]}=\frac{1}{2}$ | 0.500(3) |

from our scaling ansatz are identical to the Ginzburg criterion that ensures a "safe" distance from the nontrivial critical point of the finite-range system. The role of the Ginzberg criterion in nucleation theory was discussed by Binder [3], and recently Binder and Deutsch [5] have extended this discussion to consider finite-size crossover scaling between nontrivial and mean-field critical behavior in hypercubic systems with side length $L$ and force range $\mathcal{R}$. Their approach is to eliminate the reduced temperature from the order-parameter probability distribution along the crossover line through the Ginzburg criterion, as in our Eq. (14a), in order to obtain crossover scaling relations for the order parameter and the susceptibility in terms of $L$ and $\mathcal{R}$. As an example of how our FRS results can be obtained from their crossover scaling relations we consider the order parameter, for which they give the relation (generalized here to $\phi^n$ field theory)

$$\begin{aligned}\langle|\phi|\rangle &= L^{-\tilde{\beta}(n)D/[\tilde{\gamma}+2\tilde{\beta}(n)]} \\ &\quad\times\mathcal{M}(L^{\tilde{\nu}[D_c(n)-D]D/[\tilde{\gamma}+2\tilde{\beta}(n)]}\mathcal{R}^{-D}) \\ &= L^{-D/n}\tilde{\mathcal{M}}(L\mathcal{R}^{-D_c(n)/[D_c(n)-D]}) ,\end{aligned} \tag{44}$$

where the second line is obtained by using Eqs. (6a), (6c), and (14c) for $\tilde{\gamma}$, $\tilde{\beta}(n)$, and $D_c(n)$, respectively. To obtain critical FRS from this result, we eliminate the system size $L$ through the finite-size scaling relation $L\sim\xi$. The argument in $\tilde{\mathcal{M}}$ then yields $\xi\sim\mathcal{R}^{D_c(n)/[D_c(n)-D]}$, in agreement with our Eq. (15b), and substitution of $\mathcal{R}$ into the prefactor yields $\langle|\phi|\rangle\sim\mathcal{R}^{-2\tilde{\beta}(n)D/[D_c(n)-D]}$. The latter result can also be obtained from our Eq. (15a) by differentiation with respect to $h$. Thus our critical finite-range scaling relations can be obtained from crossover finite-size scaling by *simultaneously* requiring both the crossover and the critical scaling conditions to be satisfied. As noted by Binder and Deutsch, finite-size crossover scaling is applicable near the mean-field critical point, where hyperscaling does not hold, because the correlation length $\xi$ along the crossover line is proportional to the thermal length $l_T\propto(\chi/\phi^2)^{1/D}$ [41,42].

Renormalization groups for systems with long-range interactions have been studied in real space by Knops, van Leeuwen, and Hemmer [9] and in momentum space by Green [10] and Gunton and Yalabik [11]. These renormalization groups have a "van der Waals" fixed point with classical exponents, which becomes stable for perturbations away from infinite force range only for $D>D_c$. We now show that our FRS relations may be obtained from the results of these renormalization-group (RG) studies. We use a notation close to that of Knops, van Leeuwen, and Hemmer, but we extend our discussion to cover a general $\phi^n$ field theory, as done by Gunton and Yalabik. For a GL Hamiltonian equivalent to that of Eq. (1) one obtains the following scaling relations for the singular free-energy density $F$, the correlation length $\xi$, and the force range $\mathcal{R}$ under an RG transformation which dilutes the degrees of freedom $S$ by a factor $S'/S=l^{-D}$:

$$F(h,t,g_n)=l^{-D}F(\lambda_1 h,\lambda_2 t,\lambda_n g_n) , \tag{45a}$$

$$\xi(h,t,g_n)=l\xi(\lambda_1 h,\lambda_2 t,\lambda_n g_n) , \tag{45b}$$

$$\mathcal{R}=\lambda_{\mathcal{R}}^{-1}\mathcal{R}' , \tag{45c}$$

where the eigenvalues are

$$\lambda_i=l^{D-i\tau} , \tag{46a}$$

$$\lambda_{\mathcal{R}}=l^{(D/2)-\tau-1} . \tag{46b}$$

The crucial point of this RG calculation is to keep the nonlinearity $g_n$ constant by choosing $\tau=D/n$ to yield $\lambda_n=1$ and

$$\lambda_1=l^{D(1-1/n)}=l^{D\tilde{\Delta}(n)/[\tilde{\nu}D_c(n)]} , \tag{47a}$$

$$\lambda_2=l^{D(1-2/n)}=l^{D/[\tilde{\nu}D_c(n)]} , \tag{47b}$$

$$\lambda_{\mathcal{R}}=l^{D[(1/2)-(1/n)]-1}=l^{-[D_c(n)-D]/[\tilde{\nu}D_c(n)]} , \tag{47c}$$

where $\tilde{\nu}$, $\tilde{\Delta}(n)$, and $D_c(n)$ are given by Eqs. (10a), (10b), and (14c). The scaling relation Eq. (45c) for $\mathcal{R}$ and the explicit relation Eq. (47c) for its eigenvalue allow us to eliminate $l$ from the scaling relations for $F$ and $\xi$, yielding

$$F(h,t,g_n)=(\mathcal{R}'/\mathcal{R})^{DD_c(n)/\{\tilde{\nu}[D_c(n)-D]\}}F((\mathcal{R}'/\mathcal{R})^{-D\tilde{\Delta}(n)/\{\tilde{\nu}[D_c(n)-D]\}}h,(\mathcal{R}'/\mathcal{R})^{-D/\{\tilde{\nu}[D_c(n)-D]\}}t,g_n) , \tag{48a}$$

$$\xi(h,t,g_n)=(\mathcal{R}'/\mathcal{R})^{-D_c(n)/\{\tilde{\nu}[D_c(n)-D]\}}\xi((\mathcal{R}'/\mathcal{R})^{-D\tilde{\Delta}(n)/\{\tilde{\nu}[D_c(n)-D]\}}h,(\mathcal{R}'/\mathcal{R})^{-D/\{\tilde{\nu}[D_c(n)-D]\}}t,g_n) , \tag{48b}$$

which are equivalent to our FRS scaling relations of Eqs. (15a)–(15d). In these calculations the $n$th-order nonlinearity, which is a dangerous irrelevant variable above $D_c$ [30], was kept constant by a judicious choice of the spin-rescaling parameter $\tau$ in the RG. In the present work this control is achieved by the scaling of the order parameter on its zero-temperature value as discussed in Sec. II.

A question not discussed in detail in Sec. III is the possibility of logarithmic corrections to the FRS relations, which are known to be present for the similar finite-*size* scaling of short-range systems at their upper critical dimension [15]. This point has been considered at length in earlier work on FRS, both for the infinitely coordinated models studied by Botet *et al.* [16,17], and for the one-dimensional model with power-law interactions studied by Glumac and Uzelac [18,19]. In all cases studied by these authors, pure power-law scaling was observed numerically to considerable accuracy, with no signs of logarithmic corrections. For the infinitely coordinated Ising model the magnetization was obtained analytically by Botet and Jullien [17], verifying the absence of logarithmic corrections in this special case. Logarithmic corrections were also found to be absent in an infinite-range mean-field percolation model studied by Privman and Schulman [43]. As was shown numerically in Sec. IV, we find that the result is the same for the Q1DI model: the power-law FRS relations derived in Sec. III C were verified to high numerical accuracy, and the exponential improvement on the convergence of bulk exponent estimates due to filtering of successive powers indicated that no logarithmic corrections were present.

## VI. DISCUSSION

In this paper we have derived critical scaling relations for general $D$-dimensional $\phi^n$ scalar-field theories, and we have shown that the scaling ansatz for the singular free-energy density, on which this derivation is based, is equivalent to a Ginzburg criterion. By comparing these finite-range scaling relations to finite-*size* scaling results for cylindrical systems with $D > D_c$ we have identified a condition of *critical equivalence* under which the two kinds of systems can be asymptotically mapped onto one another at the mean-field critical point. We have applied these scaling relations to a standard $\phi^4$ GL Hamiltonian, to the one-dimensional Kac model with exponentially decaying ferromagnetic interactions, and to the ferromagnetic quasi-one-dimensional Ising (Q1DI) model. We find that near the Gaussian mean-field critical point the Ginzburg-Landau Hamiltonians for all three models become identical. However, for the Q1DI model a length rescaling is required to obtain this mapping, leading to different scaling relations for the correlation lengths. The scaling relations obtained for the Kac model are those of a one-dimensional quartic field theory, whereas for the Q1DI model they are those of a cylindrical Ising system with $D > D_c$, supporting our proposed criterion of critical equivalence. We have shown further that our scaling relations can also be obtained from critical finite-*size* crossover scaling [5] and from renormalization-group results [9–11]. These findings give finite-range scaling a physical basis separate from the simple analogy with finite-*size* scaling.

We note that the critical-equivalence mapping introduced in Sec. II can be used to relate the critical behavior of a $D$-dimensional field theory with force range $\mathcal{R}$ to that of the Q1DI model ($D'=1$, $D > D_c$) with cross section $N$, which has the advantage that it can be studied easily by numerical transfer-matrix methods. The resulting mapping, obtained from Eq. (17), is

$$\mathcal{R} \sim N^{(D_c - D)/[D(D_c - 1)]} . \tag{49}$$

For $D=1$, such as in the Kac model, this gives the simple mapping $\mathcal{R} \sim N$, which was used in Sec. III. For a general dimensionality $D$ Eq. (49) can be interpreted as a statement that the Q1DI model is critically equivalent with *either* a field theory with $D < D_c$ and long-range interactions, in which $R$ is a monotonically increasing function of $N$, *or* a field theory with $D = D_c$ and interactions of arbitrary range ($R \sim N^0$), *or* a field theory with $D > D_c$ and short-range interactions, in which $R$ is a monotonically decreasing function of $N$. The latter case can be viewed as an expression of the stability of the "van der Waals" fixed point for systems with finite-range interactions and $D > D_c$. If the correspondence discussed above is interpreted literally, the Q1DI model could possibly be seen as representing transfer in a direction perpendicular to the "surface" of a critical cluster in any of the critically equivalent $D$-dimensional models.

Our analytical results for the Q1DI model were confirmed by high-precision numerical transfer-matrix calculations obtained by specialized numerical scaling techniques. These numerical results verified the absence of logarithmic corrections, indicating a marked difference of finite-range scaling from finite-size scaling of short-range systems for $D = D_c$. Further, if Eq. (49) for $D=3$ is combined with the mapping of polymer blends of finite chain length onto the three-dimensional Ising model with long-range interactions, given by Binder and Deutsch [4,5], a mapping is obtained which connects the critical chain-length scaling in the polymer to the $N$-scaling behavior of the Q1DI model. The Monte Carlo results for the polymer system presented in Ref. [5] are consistent with our numerical results for the Q1DI model. Both the theoretical and the numerical results obtained here underscore the ambiguity of the notion of "length" in mean-field-like models.

A numerical study of the Q1DI model near its spinodal line, which corresponds to a $\phi^3$ field theory [11,40], is in progress. Some preliminary results were reported in Ref. [27], and a more detailed account is planned to be reported elsewhere [44].

*Note added in proof.* A review article on finite-size scaling for systems with long-range interactions by J. G. Brankov and N. S. Tonchev [Physica A **189**, 583 (1992)] came to our attention after this paper was accepted for publication. This article discusses mainly systems with integrable power-law interactions, but also contains additional useful references to the original literature for equivalent-neighbor models.


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